



# Quiver Poisson algebras <sup>☆</sup>

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## Abstract

Inner Poisson algebras on a given associative algebra are introduced and characterized, which gives a way of constructing non-commutative Poisson structures. Applying these to the finite-dimensional path algebras  $k\vec{Q}$ , together with the decomposition into indecomposable Lie ideals of the standard Poisson structure on  $k\vec{Q}$ , we classify all the inner Poisson structures on  $k\vec{Q}$ , which turn out to be the piecewise standard Poisson algebras. We also determine all the finite quivers  $\vec{Q}$  without oriented cycles such that  $k\vec{Q}$  admits outer Poisson structures: these are exactly the finite quivers without oriented cycles such that there exist two non-trivial paths  $\alpha$  and  $\beta$  lying in a reduced closed walk, which cannot be connected by a sequence of non-trivial paths.

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## Introduction

The aim of this paper is to construct finite-dimensional, non-commutative Poisson algebras via quivers. Here by a *Poisson algebra* over a field  $k$  we mean a triple  $(A, \cdot, \{-, -\})$ , where  $(A, \cdot)$  is an associative  $k$ -algebra and  $(A, \{-, -\})$  is a Lie  $k$ -algebra, such that the Leibniz rule  $\{a, bc\} = \{a, b\}c + b\{a, c\}$  holds for  $a, b, c \in A$  (or equivalently,  $\{ab, c\} = a\{b, c\} + \{a, c\}b$  holds

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for  $a, b, c \in A$ ). We stress that there are other different definitions for non-commutative Poisson algebras, see e.g. Xu [X], Definition 1.1 in Crawley-Boevey [C], and Definition 2.6.1 in Van den Bergh [Van]. For details see Remark 3.8.

Let  $(A, \cdot, \{-, -\})$  be a Poisson algebra. For  $a \in A$ , denote the Hamiltonian of  $a$  by  $\text{ham}(a) = \{a, -\} \in \text{End}_k(A, A)$ . Then the Leibniz rule just says that  $\text{ham}(a)$  is a derivation of the associative algebra  $(A, \cdot)$ .

Let  $(A, \cdot)$  be an associative algebra with  $a, b \in A$ . Denote by  $[a, b]$  the commutator  $ab - ba$  of  $a$  and  $b$ . Then for any  $\lambda \in k$ ,  $(A, \cdot, \lambda[-, -])$  is a Poisson algebra, which is called a *standard Poisson structure* on  $(A, \cdot)$ .

Non-commutative Poisson algebras are widely used in non-commutative geometry and mathematical physics (see e.g. [DB,K,L,V], and [X]). However, there is a lack of examples of non-commutative, non-standard Poisson algebras (see e.g. Farkas–Letzter [FL, p. 157]). On the other hand, any Poisson structures on some classes of non-commutative associative algebras are known to be standard, for examples, on the simple algebras (Kubo [Ku1]), on the algebras  $T_n(k)$  of upper triangular matrices [Ku1], on the poset subalgebras of  $M_\infty(\mathbb{C})$  [Ku2], and on the non-commutative prime algebras [FL]. Also, it is proved in [Ku1] that the associative product in any Poisson structure on a semisimple Lie algebra is trivial. From the algebraic viewpoint, it is then natural to construct non-commutative, non-standard Poisson algebras. Inspired by the recent works of Bocklandt and Le Bruyn [BL], Crawley-Boevey, Etingof and Ginzburg [CEG], and Van den Bergh [Van], in this paper we will deal with this construction via the quiver techniques by considering the so-called inner Poisson algebras and outer-Poisson quivers, though we do not know how to induce commutative Poisson structures from the non-commutative ones given in this paper.

A Poisson algebra  $(A, \cdot, \{-, -\})$  is said to be *inner* if  $\text{ham}(a)$  is an inner derivation of  $(A, \cdot)$  (i.e.,  $\text{ham}(a) = [a', -]$  for some  $a' \in A$ ) for each  $a \in A$ . As a natural generalization of standard Poisson algebras, inner Poisson structures often arise: if the first Hochschild cohomology of  $(A, \cdot)$  vanishes, then any Poisson structure on  $(A, \cdot)$  is inner (see Gerstenhaber [G]). A theorem due to Happel [H] says that the first Hochschild cohomology of a finite-dimensional path algebra  $k\vec{Q}$  vanishes if and only if  $\vec{Q}$  is a finite tree. It follows that any Poisson structure on the path algebra of a finite tree is inner (however the converse is not true. See Example 4.3(i)).

This paper is organized as follows. In Section 1 we characterize the isoclasses of inner Poisson algebras on a given associative algebra  $A$  by the equivalence classes of  $\mathcal{P}(A)$ , a special class of linear transformations of  $A$  as defined in 1.3. See Theorem 1.4. With this characterization we construct new inner Poisson algebras from the given one (Proposition 1.9).

In order to classify all the inner Poisson structures on a finite-dimensional path algebra  $k\vec{Q}$ , and to determine all the finite quivers  $\vec{Q}$  without oriented cycles such that  $k\vec{Q}$  admits outer Poisson structures, we need to decompose the standard Lie structure on  $k\vec{Q}$  into indecomposable Lie ideals. This is done in Section 2, by introducing an equivalence relation on the set of non-trivial paths of  $\vec{Q}$ , and a symmetric bilinear form  $\langle -, - \rangle$  on  $k\vec{Q}_0$  such that each subspace of  $k\vec{Q}_0$  has an orthogonal basis with respect to  $\langle -, - \rangle$ , where  $k\vec{Q}_0$  is the  $k$ -space spanned by the vertices of  $\vec{Q}$ . See Theorem 2.10.

By a *piecewise standard Poisson algebra*  $(A, \cdot, \{-, -\}, \lambda_1, \dots, \lambda_m)$  on a given associative algebra  $(A, \cdot)$  we mean that there exists a decomposition  $A = \bigoplus_{1 \leq i \leq m} A_i$  into indecomposable ideals of the standard Lie algebra  $(A, [-, -])$ , and a vector  $(\lambda_1, \dots, \lambda_m) \in k^m$ , such that  $\text{ham}(a) = \lambda_i[a, -]$  for each  $a \in A_i$ ,  $1 \leq i \leq m$ . It is proved in Section 3 that any inner Poisson structure on a finite-dimensional path algebra  $k\vec{Q}$  is piecewise standard; and conversely, given an arbitrary vector  $(\lambda_1, \dots, \lambda_m) \in k^m$ , where  $m$  is the degree of the quiver  $\vec{Q}$  (cf. 2.2), there is a

unique piecewise standard Poisson algebra  $(k\vec{Q}, \cdot, \{-, -\})$  such that  $\text{ham}(a) = \lambda_i[a, -]$  for each  $a \in A_i, 1 \leq i \leq m$ . See Theorem 3.5.

Although most of Sections 2 and 3 holds in a more general setup (see Remarks 2.7 and 3.7), for the simplicity we still choose quivers to state our results. We also emphasize that an inner Poisson algebra is not necessarily a piecewise standard Poisson algebra. See Example 3.6(i).

In the final section, we determine all the finite quivers  $\vec{Q}$  without oriented cycles such that  $k\vec{Q}$  admits outer Poisson structures, or equivalently, all the finite quivers  $\vec{Q}$  without oriented cycles such that any Poisson structure on  $k\vec{Q}$  is inner: these are exactly the finite quivers without oriented cycles such that any two non-trivial paths  $\alpha$  and  $\beta$  lying in a reduced closed walk can be connected by a sequence of non-trivial paths (cf. 2.2). See Theorem 4.2.

Throughout this paper,  $k$  is a field of characteristic 0. For unexplained notions on quivers we refer to Auslander–Reiten–Smalø [ARS] and Ringel [R].

### 1. Inner Poisson algebras

In a Poisson algebra  $(A, \cdot, \{-, -\})$ , denote by  $Z(A)$  and  $Z\{A\}$  the centers of the Lie bracket  $[-, -]$  and  $\{-, -\}$ , respectively. Denote by  $[A, A]$  and  $\{A, A\}$  the  $k$ -subspaces spanned by all the commutators  $[a, b]$ , and by all the elements  $\{a, b\}$ , respectively, where  $a, b$  run over  $A$ . Note that  $Z(A)$  is exactly the center of the associative algebra  $(A, \cdot)$ . We need the following easy fact.

**Lemma 1.1.** *Let  $(A, \cdot, \{-, -\})$  be an inner Poisson algebra. Then  $Z(A) \subseteq Z\{A\}$ , and  $\{A, A\} \subseteq [A, A]$ .*

**Proof.** Let  $a \in Z(A)$ . For any  $x \in A$  there exists  $x' \in A$  such that  $\text{ham}(x) = [x', -]$ . It follows that

$$\{a, x\} = -\{x, a\} = -[x', a] = 0$$

which implies  $a \in Z\{A\}$ .  $\square$

**Lemma 1.2.** *Let  $(A, \cdot)$  be an associative algebra. Then any inner Poisson algebra  $(A, \cdot, \{-, -\})$  on  $(A, \cdot)$  is given by  $\text{ham}(a) = [g(a), -], \forall a \in A$ , where  $g$  is a  $k$ -linear transformation of  $A$  satisfying*

$$[g(x), y] = [x, g(y)], \quad \forall x, y \in A, \tag{1.1}$$

$$[g(x), g(y)] - g([g(x), y]) \in Z(A), \quad \forall x, y \in A, \tag{1.2}$$

and

$$Z(A) \subseteq \text{Ker}(g). \tag{1.3}$$

*Conversely, if  $g$  is a  $k$ -linear transformation of  $A$  satisfying (1.1) and (1.2), then  $(A, \cdot, \{-, -\})$  is an inner Poisson algebra, where  $\text{ham}(a) = [g(a), -]$  for each  $a \in A$ .*

**Proof.** Assume that  $(A, \cdot, \{-, -\})$  is an inner Poisson algebra. Then for each  $a \in A$  there exists  $a' \in A$  such that  $\text{ham}(a) = [a', -]$ . If  $[a', -] = [a'', -]$  then  $a' - a'' \in Z(A)$ . This permits us to

define a  $k$ -linear map  $\tilde{g} : A \rightarrow A/Z(A)$  by  $\tilde{g}(a) = \pi(a')$ , where  $\pi : A \rightarrow A/Z(A)$  is the canonical projection.

For  $a \in Z(A)$ , by Lemma 1.1 we have  $\text{ham}(a) = 0$ , and hence  $\tilde{g}(a) = 0$  by definition. It follows that we can choose a lift  $g$  of  $\tilde{g}$ , i.e. a linear map  $g : A \rightarrow A$  with  $\pi g = \tilde{g}$ , such that  $Z(A) \subseteq \text{Ker}(g)$ . (In fact, let  $B \cup C$  be a basis of  $A$  such that  $B$  is a basis of  $Z(A)$  and  $B \cap C = \emptyset$ . Take a map  $g : B \cup C \rightarrow A$  such that  $g(B) = 0$  and  $\pi g(x) = \tilde{g}(x)$  for  $x \in C$ . Then we are done by extending  $g$  linearly.)

Since  $a' - g(a) \in \text{Ker } \pi = Z(A)$ , it follows that  $\text{ham}(a) = [a', -] = [g(a), -], \forall a \in A$ . Then we have

$$[g(x), y] = \{x, y\} = -\{y, x\} = -[g(y), x] = [x, g(y)].$$

It remains to prove (1.2). For those algebras  $A$  where  $[A, A] \subseteq Z(A)$ , (1.2) follows from (1.3). In general, by the Jacobi identity  $\{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0$  we have

$$g([\{g(x), y\}, z]) + [[\{g(y), z\}, g(x)] + [[\{z, g(x)\}, g(y)]] = 0.$$

By the Jacobi identity of the bracket  $[-, -]$  we have

$$g([\{g(x), y\}, z]) = [[\{g(x), g(y)\}, z].$$

Since  $z$  is arbitrary, it follows that

$$g([\{g(x), y\}]) - [g(x), g(y)] \in Z(A).$$

Conversely, if  $g$  is a  $k$ -linear transformation of  $A$  satisfying (1.1) and (1.2), then it is easy to verify that  $(A, \{-, -\})$  given by  $\text{ham}(a) = [g(a), -]$  for each  $a \in A$ , is a Lie algebra: (1.1) implies the anti-symmetry, and (1.2) implies the Jacobi identity. It is clear that  $(A, \cdot, \{-, -\})$  is a Poisson algebra.  $\square$

**1.3.** Let  $(A, \cdot)$  be an associative algebra. Set  $\mathcal{P}(A)$  to be the set of the  $k$ -linear transformations  $g$  of  $A$  satisfying (1.1)–(1.3). Define a relation  $\sim$  on  $\mathcal{P}(A)$ :  $g \sim g'$  if and only if there exists  $\tau \in \text{Aut}(A, \cdot)$  such that  $\text{Im}(\tau g \tau^{-1} - g') \subseteq Z(A)$ , where  $\text{Aut}(A, \cdot)$  is the automorphism group of the associative algebra  $(A, \cdot)$ . It is easy to see that this is an equivalence relation on  $\mathcal{P}(A)$ . Denote by  $[g]$  the equivalence class of  $g$ .

Two Poisson structures on  $(A, \cdot)$  are said to be isomorphic as Poisson algebras provided that there exists an associative algebra automorphism  $\tau$  of  $(A, \cdot)$  such that  $\tau$  is also a Lie algebra homomorphism. Denote by  $[(A, \cdot, \{-, -\})]$  the isoclass of a Poisson algebra  $(A, \cdot, \{-, -\})$ .

**Theorem 1.4.** *Let  $(A, \cdot)$  be an associative algebra. Then the map*

$$\{ \text{the equivalence class of } \mathcal{P}(A) \} \rightarrow \{ \text{the isoclass of inner Poisson structure on } (A, \cdot) \}$$

given by

$$[g] \mapsto [(A, \cdot, \{-, -\})], \quad \text{where } \text{ham}(a) = [g(a), -], \quad \forall a \in A,$$

is bijective.

**Proof.** The map given above is well-defined and injective: If  $g, g' \in \mathcal{P}(A)$ , then  $[g] = [g']$  if and only if there exists  $\tau \in \text{Aut}(A, \cdot)$  such that  $\text{Im}(\tau g \tau^{-1} - g') \subseteq Z(A)$ , if and only if there exists  $\tau \in \text{Aut}(A, \cdot)$  such that  $\tau : (A, \{-, -\}) \rightarrow (A, \{-, -\}')$  is also a Lie algebra isomorphism, where  $\{a, b\} = [g(a), b]$  and  $\{a, b\}' = [g'(a), b]$  for  $a, b \in A$ .

It follows from Lemma 1.2 that the map given above is also surjective.  $\square$

**Remark 1.5.** (i) If  $g \in \mathcal{P}(A)$  then  $\tau g \tau^{-1} \in \mathcal{P}(A)$  for any  $\tau \in \text{Aut}(A, \cdot)$ . If  $g, g' \in \mathcal{P}(A)$  and  $[g] = [g']$ , then  $[\tau g \tau^{-1}] = [\tau g' \tau^{-1}]$  for any  $\tau \in \text{Aut}(A, \cdot)$ . By Theorem 1.4 this implies that the group  $\text{Aut}(A, \cdot)$  has a left action on the set of the isoclasses of inner Poisson structures on  $(A, \cdot)$  by conjugation.

(ii) Theorem 1.4 permits us to write an inner Poisson algebra  $(A, \cdot, \{-, -\})$  on  $(A, \cdot)$  by  $(A, \cdot, g)$  with  $g \in \mathcal{P}(A)$ .

Assume that  $g, h \in \mathcal{P}(A)$ . If  $gh = hg$  then  $gh \in \mathcal{P}(A)$ , and hence we have the inner Poisson algebra  $(A, \cdot, gh)$ . If  $g([h(x), y]) = [h(x), g(y)]$  and  $h([g(x), y]) = [g(x), h(y)]$  for  $x, y \in A$ , then  $g + h \in \mathcal{P}(A)$ , and hence we have the inner Poisson algebra  $(A, \cdot, g + h)$ .

(iii) Denote by  $\text{Aut}(A, \cdot, g)$  the automorphism group of the inner Poisson algebra  $(A, \cdot, g)$ , where  $g \in \mathcal{P}(A)$ . Then  $\text{Aut}(A, \cdot, g) = \{\tau \in \text{Aut}(A, \cdot) \mid \text{Im}(\tau g \tau^{-1} - g) \subseteq Z(A)\}$ .

(iv) Let  $(A, \cdot, g)$  be an inner Poisson algebra with  $g \in \mathcal{P}(A)$ . Since  $Z(A) \subseteq Z\{A\}$ , it follows that on  $\bar{A} := A/Z(A)$  there are two Lie algebra structures  $(\bar{A}, \{-, -\})$  and  $(\bar{A}, [-, -])$ .

Since  $Z(A) \subseteq \text{Ker}(g)$ , it follows that  $g$  induces a  $k$ -linear map  $\hat{g} : \bar{A} \rightarrow A$  by  $\hat{g}\pi = g$ . Define  $\bar{g} := \pi \hat{g} \in \text{End}_k(\bar{A})$ . Then  $\bar{g}(\bar{x}) = g(x), \forall x \in A$ , where  $\bar{x} = \pi(x)$ . By (1.1) and (1.2) we deduce that

$$[\bar{g}(\bar{x}), \bar{y}] = [\bar{x}, \bar{g}(\bar{y})], \quad \bar{g}([\bar{g}(\bar{x}), \bar{y}]) = [\bar{g}(\bar{x}), \bar{g}(\bar{y})], \quad \forall \bar{x}, \bar{y} \in \bar{A};$$

and that  $\bar{g} : (\bar{A}, \{-, -\}) \rightarrow (\bar{A}, [-, -])$  is a Lie algebra homomorphism:

$$\bar{g}(\{\bar{x}, \bar{y}\}) = \bar{g}(\overline{\{x, y\}}) = \bar{g}(\overline{[g(x), y]}) = \overline{[g(x), y]} = \overline{[g(x), g(y)]} = [\bar{g}(\bar{x}), \bar{g}(\bar{y})].$$

(v) Let  $(A, \cdot)$  be an associative algebra. Denote by  $\text{Der}(A)$  and  $\text{Inn}(A)$  the space of the derivations and the inner derivations of  $A$ , respectively. Then  $\text{Inn}(A)$  is a subalgebra of the Lie algebra  $(\text{Der}(A), [-, -])$ , and  $\text{Inn}(A) \cong (\bar{A}, [-, -])$  as Lie algebras.

If  $(A, \cdot, \{-, -\})$  is a Poisson algebra, then the Hamiltonian map  $\text{ham} : A \rightarrow \text{Der}(A)$  gives a Lie algebra homomorphism from  $(A, \{-, -\})$  to  $(\text{Der}(A), [-, -])$ . If  $(A, \cdot, \{-, -\})$  is an inner Poisson algebra given by  $(A, \cdot, g)$  with  $g \in \mathcal{P}(A)$ , then the Hamiltonian map  $\text{ham}$  gives a Lie algebra homomorphism from  $(A, \{-, -\})$  to  $\text{Inn}(A)$ , and hence to  $(\bar{A}, [-, -])$ . This exactly says that the map  $\bar{g} \in \text{End}_k(\bar{A})$  defined in (iv) satisfies  $\bar{g}([\bar{g}(x), \bar{y}]) = [\bar{g}(\bar{x}), \bar{g}(\bar{y})]$ .

**1.6.** In the rest part of this section we fix the following notations. Let  $(A, \cdot)$  be a finite-dimensional associative algebra, and  $g \in \mathcal{P}(A)$ . Consider the generalized eigenspace decomposition of  $g$ . Let  $\{\lambda_0, \dots, \lambda_m\}$  be the set of eigenvalues of  $g$ , and  $V_i$  be the corresponding root space

$$\{x \in A \mid (g - \lambda_i)^s(x) = 0, \text{ for some } s \geq 0\}.$$

Then  $V_i$  is a  $g$ -invariant subspace and  $A = \bigoplus_{0 \leq i \leq m} V_i$ . Since  $0 \neq 1 \in Z(A) \subseteq \text{Ker}(g)$ , it follows that we may assume that  $\lambda_0 = 0$ , and hence  $Z(A) \subseteq V_0$ .

**Lemma 1.7.** *We have*

- (i)  $[V_i, V_j] = 0$ , for  $i \neq j$ .
- (ii)  $[V_0, V_0] \subseteq V_0$ , and hence  $V_0$  is an ideal of the Lie algebra  $(A, [-, -])$ .
- (iii)  $[V_i, V_i] \subseteq Z(A) \oplus V_i$  for  $i \neq 0$ , and hence  $Z(A) \oplus V_i$  is an ideal of the Lie algebra  $(A, [-, -])$  for  $i \neq 0$ .

**Proof.** (i) Let  $x \in V_i, y \in V_j$ . By applying (1.1) one has  $[g^s(x), y] = [x, g^s(y)]$  for all  $s \geq 0$ . By definition  $(g - \lambda_i)^t(x) = 0$  for some  $t$ . Since  $\lambda_i \neq \lambda_j$  for  $i \neq j$ , it follows that  $g - \lambda_i$  is invertible on  $V_j$ , and hence  $(g - \lambda_i)^s$  is invertible on  $V_j$  for any  $s \geq 0$ . It follows that there exists  $y' \in V_j$  such that  $y = (g - \lambda_i)^t(y')$ . Thus

$$[x, y] = [x, (g - \lambda_i)^t(y')] = [(g - \lambda_i)^t(x), y'] = 0.$$

(ii) and (iii). Let  $x, y \in V_i$ . Write  $[x, y] = \sum_{0 \leq j \leq m} x_j$  with  $x_j \in V_j$ . For each  $j \neq i$ , by the Jacobi identity and (i) we have

$$[[x, y], V_j] \subseteq [[x, V_j], y] + [x, [y, V_j]] = 0.$$

While

$$[[x, y], V_j] = \left[ \sum_{0 \leq s \leq m} x_s, V_j \right] = [x_j, V_j].$$

It follows that  $[x_j, V_j] = 0$  for  $j \neq i$ , and then by (i) we have  $[x_j, A] = 0$  for  $j \neq i$ . This implies  $x_j \in Z(A) \cap V_j$  for  $j \neq i$ .

If  $i = 0$  then  $x_j \in Z(A) \cap V_j$  for  $j \neq 0$ . Thus  $x_j = 0$  for  $j \neq 0$ . This proves that  $[V_0, V_0] \subseteq V_0$ .

If  $i \neq 0$  then  $x_j \in Z(A) \cap V_j$  for  $j \neq i$ . Thus  $x_j = 0$  for  $j \neq i, j \neq 0$ , i.e.  $[x, y] = x_0 + x_i$  with  $x_0 \in Z(A)$ . This proves that  $[V_i, V_i] \subseteq Z(A) \oplus V_i$ .  $\square$

**Remark 1.8.** In general,  $V_i$  ( $i \neq 0$ ) is not an ideal of the Lie algebra  $(A, [-, -])$ . For example, let  $A = k\langle x, y \rangle / \langle x, y \rangle^3$ . Then  $Z(A)$  is the space spanned by 1 and all the monomials of total degree 2. Denote by  $V_1$  the subspace spanned by  $\bar{x}$  and  $\bar{y}$ . Let  $g$  be the  $k$ -linear transformation of  $A$  given by  $g|_{Z(A)} = 0$  and  $g|_{V_1} = \text{Id}$ . Then  $A = Z(A) \oplus V_1$  is the decomposition of root spaces of  $g$ . It is easy to see  $g \in \mathcal{P}(A)$ . Note that  $[V_1, V_1] \subseteq Z(A)$  is the root space decomposition of  $g$ .

**Proposition 1.9.** *Keep the notations in 1.6. Let  $f_0(t), \dots, f_m(t)$  be polynomials such that  $f_0(0) = 0$ . Then the  $k$ -linear transformation defined by*

$$f(g) : \bigoplus_{0 \leq i \leq m} V_i \rightarrow \bigoplus_{0 \leq i \leq m} V_i, \quad f(g)|_{V_i} = f_i(g)$$

*belongs to  $\mathcal{P}(A)$ , and hence  $f(g)$  induces an inner Poisson structure on  $(A, \cdot)$ .*

**Proof.** Set  $p := f(g)$ . Since  $f_0(0) = 0$ , it follows from the construction that  $p(Z(A)) = 0$ . Since  $A = \bigoplus_{0 \leq i \leq m} V_i$  is a decomposition of invariant subspaces of  $p$ , and  $[V_i, V_j] = 0$  for  $i \neq j$ , it suffices to prove that  $p|_{V_i}$  satisfies (1.1) and (1.2).

Assume that  $x, y \in V_i$ . Since  $g$  satisfies (1.1), it follows that  $p|_{V_i}$  also satisfies (1.1):

$$[p(x), y] = [f_i(g)(x), y] = [x, f_i(g)(y)] = [x, p(y)].$$

If  $i \neq 0$  then  $\lambda_i \neq 0$ , and hence  $g|_{V_i}$  is invertible. It follows that  $x = g(x')$  for some  $x' \in V_i$ , and then by (1.2) we have

$$g([x, y]) = g([g(x'), y]) = [g(x'), g(y)] + z = [x, g(y)] + z$$

for some  $z \in Z(A)$ . Applying this identity iteratively we have

$$p([p(x), y]) = f_i(g)([p(x), y]) = [p(x), f_i(g)(y)] + z' = [p(x), p(y)] + z'$$

for some  $z' \in Z(A)$ . This shows that  $p|_{V_i}$  satisfies (1.2) for  $i \neq 0$ .

By assumption  $f_0(t) = tu(t)$  for some polynomial  $u(t)$ . It follows from (1.2) that

$$\begin{aligned} p([p(x), y]) &= f_i(g)([g(u(g)(x)), y]) \\ &= [g(u(g)(x)), f_i(g)(y)] + z \\ &= [p(x), p(y)] + z \end{aligned}$$

for some  $z \in Z(A)$ . This proves that  $p|_{V_0}$  also satisfies (1.2).

By Theorem 1.4  $f(g)$  induces an inner Poisson structure on  $(A, \cdot)$ .  $\square$

**Remark 1.10.** If one can choose polynomials  $f_i(t)$  in Proposition 1.9 such that  $g\tau f(g)\tau^{-1} - g^2 \neq 0$  for any  $\tau \in \text{Aut}(A, \cdot)$ , then  $\text{Im}(\tau f(g)\tau^{-1} - g) \not\subseteq Z(A)$  for any  $\tau \in \text{Aut}(A, \cdot)$ , and hence by Theorem 1.4 the inner Poisson structure on  $(A, \cdot)$  induced by  $f(g)$  is not isomorphic to the one induced by  $g$ . In this way one obtains new inner Poisson algebras from the known one.

## 2. Standard Lie structure on path algebras

As we will see in the next section, the Lie ideals of an inner Poisson structure on a finite-dimensional path algebra  $k\vec{Q}$  are exactly the ones of the standard Lie algebra on  $(k\vec{Q}, [-, -])$ . The aim of this section is to decompose the standard Lie algebra  $(k\vec{Q}, [-, -])$  into a direct sum of indecomposable Lie ideals. This is needed in classifying all the inner Poisson structures on  $k\vec{Q}$  in Section 3, and in determining all the outer-Poisson quivers in Section 4.

### 2.1. For the quiver technique of algebras we refer to [ARS] and [R].

Recall that a quiver  $\vec{Q} = (\vec{Q}_0, \vec{Q}_1, s, t)$  is an oriented graph, where  $\vec{Q}_0$  is the set of vertices,  $\vec{Q}_1$  is the set of arrows, and for any arrow  $\alpha$ ,  $s(\alpha)$  and  $t(\alpha)$  are the starting and ending vertex of  $\alpha$ , respectively. Let  $k\vec{Q}$  be the vector space with basis the set of all the paths in  $\vec{Q}$ . Then  $k\vec{Q}$  is finite-dimensional if and only if  $\vec{Q}$  is a finite quiver without oriented cycles, and  $k\vec{Q} = \bigoplus_{n \geq 0} k\vec{Q}_n$  is a graded associative algebra with  $Z(k\vec{Q}) = k \cdot 1$ , which is called the path algebra of  $\vec{Q}$ , where the multiplication is given by the conjunction of paths, and  $k\vec{Q}_n$  is the  $k$ -space with basis the set of paths of length  $n$ . Vertex  $i \in \vec{Q}_0$  is regarded as a path of length 0 and denoted by  $e_i \in k\vec{Q}$ . A path of length  $\geq 1$  is called a *non-trivial path*. We write the conjunction of paths from right to left.

Throughout this section we assume that  $\vec{Q}$  is a finite connected quiver without oriented cycles. Thus  $1 = \sum_{i \in \vec{Q}_0} e_i$  with  $e_i e_j = \delta_{i,j}$ , where  $\delta_{i,j}$  is the Kronecker symbol.

If the orientation of  $\vec{Q}$  is forgotten, then we get the underlying graph  $Q$  of  $\vec{Q}$ . For an unoriented graph  $Q$  denote by  $(b|a)$  an edge with vertices  $a$  and  $b$ ; denote by  $(a_{m+1}|\alpha_m|a_m|\cdots|a_3|\alpha_2|a_2|\alpha_1|a_1)$  a walk, where  $\alpha_i$  is an edge with vertices  $a_i$  and  $a_{i+1}$ ,  $1 \leq i \leq m$ ,  $m \geq 1$ , such that  $a_i \neq a_{i+1}$ . By a closed walk we mean a walk  $(a_{m+1}|\alpha_m|a_m|\cdots|a_3|\alpha_2|a_2|\alpha_1|a_1)$  such that  $a_1 = a_{m+1}$ . By a reduced closed walk we mean a closed walk  $(a_1|\alpha_m|a_m|\cdots|a_3|\alpha_2|a_2|\alpha_1|a_1)$  such that the vertices  $a_1, \dots, a_m$  are pairwise different.

**2.2.** Two non-trivial paths  $\alpha$  and  $\beta$  are said to be *connected* provided that there exists a path  $\gamma$  such that  $\alpha$  and  $\beta$  are sub-paths of  $\gamma$ . For two non-trivial paths  $\alpha$  and  $\beta$ , define  $\alpha \check{\sim} \beta$  if and only if there exist non-trivial paths  $\gamma_0 = \alpha, \gamma_1, \dots, \gamma_t, \gamma_{t+1} = \beta$ , such that for each  $0 \leq i \leq t$ , either  $\gamma_i$  and  $\gamma_{i+1}$  are connected, or both  $\gamma_i$  and  $\gamma_{i+1}$  lie in a reduced closed walk of the underlying graph  $Q$  of  $\vec{Q}$ . If  $\alpha \check{\sim} \beta$  then we say that  $\alpha$  and  $\beta$  can be connected by a sequence of non-trivial paths. It is clear that  $\check{\sim}$  is an equivalence relation on the set of non-trivial paths of  $\vec{Q}$ . Let  $\{P_1, \dots, P_m\}$  be the set of the equivalence classes. We call  $m$  the degree of the quiver  $\vec{Q}$ . Regard  $P_i$  as a sub-quiver of  $\vec{Q}$ . Denote by  $V_i$  the set of vertices of  $P_i$ .

**2.3.** Set

$$E := \left\{ \sum_{i \in \vec{Q}_0} \lambda_i e_i \in k\vec{Q}_0 \mid \sum_{i \in \vec{Q}_0} \lambda_i = 0 \right\} = \sum_{i,j \in \vec{Q}_0} k(e_i - e_j) = \sum_{\alpha \in \vec{Q}_1} k(e_{t(\alpha)} - e_{s(\alpha)}). \quad (2.1)$$

Define a bilinear form  $\langle -, - \rangle : k\vec{Q}_0 \times k\vec{Q}_0 \rightarrow k$  by  $\langle e_i, e_j \rangle = \delta_{i,j}$ ,  $i, j \in \vec{Q}_0$ . For each  $1 \leq i \leq m$ , set

$$E_i := k\{e_{t(\alpha)} - e_{s(\alpha)} \mid \alpha \in P_i\}$$

and

$$\begin{aligned} F_i &:= \{x \in E \mid \langle x, E_j \rangle = 0, \forall j \neq i\} \\ &= \left\{ \sum_{v \in \vec{Q}_0} c_v e_v \mid \sum_{v \in \vec{Q}_0} c_v = 0, c_{t(\alpha)} = c_{s(\alpha)}, \forall \alpha \in P_j, \forall j \neq i \right\}. \end{aligned}$$

Then

$$F_i = \{x \in E \mid [x, \alpha] = 0, \forall \alpha \in P_j, \forall j \neq i\}. \quad (2.2)$$

Since  $k$  is assumed to be of characteristic 0, it follows that  $\mathbb{Q} \subseteq k$ , where  $\mathbb{Q}$  is the field of rational numbers. Define

$$E_{\mathbb{Q}} := \left\{ \sum_{i \in \vec{Q}_0} \lambda_i e_i \in \mathbb{Q}\vec{Q}_0 \mid \sum_{i \in \vec{Q}_0} \lambda_i = 0 \right\}.$$

Similarly we have the subspaces  $(E_i)_{\mathbb{Q}}$  and  $(F_i)_{\mathbb{Q}}$  of  $E_{\mathbb{Q}}$ . The reason to consider  $E_{\mathbb{Q}}$  is as follows. If  $x \in E_{\mathbb{Q}}$ , then  $\langle x, x \rangle = 0$  if and only if  $x = 0$ . It follows from the Gram–Schmidt



orthogonalization that we have an orthogonal basis  $\{a_1, \dots, a_s, \dots, a_t\}$  of  $(E)_{\mathbb{Q}}$  with respect to  $\langle -, - \rangle$ , such that  $\{a_1, \dots, a_s\}$  is a basis of  $\sum_{j \neq i} (E_j)_{\mathbb{Q}}$ . In this way we deduce that  $\dim_{\mathbb{Q}}(F_i)_{\mathbb{Q}} = \dim_{\mathbb{Q}} E_{\mathbb{Q}} - \dim_{\mathbb{Q}}(\sum_{j \neq i} (E_j)_{\mathbb{Q}})$ . Since  $F_i = (F_i)_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$ , it follows that

$$\dim_k F_i = \dim_k E - \dim_k \left( \sum_{j \neq i} E_j \right).$$

**Lemma 2.4.** Assume that  $0 \neq x = \sum_{j \in V_i} \lambda_j e_j \in E_i$  for some  $i$ . Then there exist  $v_1 \neq v_2 \in V_i$  such that  $\lambda_{v_1} \neq 0$  and  $\lambda_{v_2} \neq 0$ .

**Proof.** Otherwise, there is a vertex belonging to  $E_i \subseteq E$ , then all the vertices in  $\vec{Q}$  belong to  $E$ , which contradicts (2.1).  $\square$

**Lemma 2.5.** Assume that  $\vec{Q}$  is connected. Then  $E = \bigoplus_{1 \leq i \leq m} E_i$ , and hence  $E = \bigoplus_{1 \leq i \leq m} F_i$ .

**Proof.** Assume that  $\sum_{1 \leq i \leq m} x_i = 0$  with  $x_i \in E_i$  for each  $i$ . To complete the proof, it suffices to show that  $x_i = 0$  for all  $i$ . Otherwise, say  $x_1 \neq 0$ . Since  $x_1 \in E_1$ , we may write  $x_1 = \sum_{j \in V_1} \lambda_j e_j$ . It follows that there exists  $v_1 \in V_1$  such that  $\lambda_{v_1} \neq 0$ . By Lemma 2.4 there exists  $v_2 \in V_1$  such that  $v_2 \neq v_1$  and  $\lambda_{v_2} \neq 0$ . By construction  $v_1$  and  $v_2$  can be connected by some arrows in  $P_1$  (this means that there is a walk in  $P_1$  containing  $v_1$  and  $v_2$ ).

Since  $\sum_{1 \leq i \leq m} x_i = 0$ , it follows that there exists some  $x_i$ , say  $x_2$ , such that the coefficient of  $e_{v_2}$  in  $x_2$  is non-zero. Then  $v_2 \in V_2$ . Again by Lemma 2.4 one can find some  $v_3 \in V_2$  such that  $v_3 \neq v_2$  and the coefficient of  $e_{v_3}$  in  $x_2$  is non-zero. By construction  $v_2$  and  $v_3$  can be connected by some arrows in  $P_2$ .

Repeating this process we get a sequence of vertices:  $v_1 \neq v_2 \neq v_3 \neq \dots$ . Since  $\vec{Q}_0$  is a finite set, it follows that there exists  $p, q$  with  $1 \leq p < q - 1$  such that  $v_p = v_q$ . Consider the vertices  $v_p, v_{p+1}, \dots, v_q$ . For each  $j$  with  $p \leq j \leq q - 1$ , by the process of the construction above one knows that  $v_j$  and  $v_{j+1}$  can be connected by arrows in some  $P_{\sigma(j)}$ , and any three subsequent vertices  $v_j, v_{j+1}, v_{j+2}$  cannot be connected by arrows in a same  $P_i$ . Since  $v_p = v_q$ , it follows that we obtain a reduced closed walk, in which arrows belonging to different  $P_i$  lie. This contradicts the definition of  $P_i$ 's. This proves the first assertion.  $\square$

**Lemma 2.6.** Let  $\vec{Q}$  be a finite connected quiver without oriented cycles. Suppose that  $k\vec{Q} = \bigoplus_{1 \leq i \leq m} L_i$  is a decomposition of ideals of the Lie algebra  $(k\vec{Q}, [-, -])$ . Then

- (i)  $k\vec{Q}_0 = \bigoplus_{1 \leq i \leq m} (k\vec{Q}_0 \cap L_i)$ ;
- (ii) Any non-trivial path is contained in some  $L_i$ .

**Proof.** (i) Consider the Pierce decomposition  $k\vec{Q} = \bigoplus_{i, j \in \vec{Q}_0} A_{i,j}$  with  $A_{i,j} := e_i(k\vec{Q}e_j)$ . Since  $\vec{Q}$  has no oriented cycles, it follows that  $A_{i,i} = ke_i$  and  $k\vec{Q}_0 = \bigoplus_{i \in \vec{Q}_0} A_{i,i}$ .

In order to prove (i), by the Pierce decomposition above it suffices to prove that for each  $n$  and any  $x = \sum_{i, j \in \vec{Q}_0} x_{i,j} \in L_n$  with  $x_{i,j} \in A_{i,j}$ , there holds  $\sum_{i \in \vec{Q}_0} x_{i,i} \in L_n$ .

Since  $L_n$  is a Lie ideal and  $x \in L_n$ , it follows that

$$[e_q, [x, e_p]] = x_{p,q} + x_{q,p} \in L_n, \quad \forall p, q, p \neq q,$$

and hence  $\sum_{i \in \vec{Q}_0} x_{i,i} \in L_n$ .

(ii) Let  $\alpha$  be a non-trivial path. Write  $e_{s(\alpha)} = \sum_i x_i$  with  $x_i \in L_i$ . By (i) we have  $x_i = \sum_j \lambda_{i,j} e_j$  for each  $i$ . Then we have

$$\alpha = [\alpha, e_{s(\alpha)}] = \sum_{i,j} \lambda_{i,j} [\alpha, e_j] = \sum_i (\lambda_{i,s(\alpha)} - \lambda_{i,t(\alpha)}) \alpha.$$

It follows that there exists  $i$  such that  $\lambda_{i,s(\alpha)} \neq \lambda_{i,t(\alpha)}$ . Thus  $(\lambda_{i,s(\alpha)} - \lambda_{i,t(\alpha)}) \alpha = [\alpha, x_i] \in L_i$ .  $\square$

**Remark 2.7.** As we see from the proof of Lemma 2.6, if  $A$  is an associative algebra and  $1 = \sum_{1 \leq i \leq m} e_i$  with  $e_i e_j = \delta_{i,j} e_i$ ,  $1 \leq i, j \leq m$ , and if  $e_i A e_i$  is of dimension one for each  $i$ , then the corresponding assertion also holds for  $A$ .

**Lemma 2.8.** Let  $\vec{Q}$  be a finite connected quiver without oriented cycles. Suppose that  $k\vec{Q} = \bigoplus_{1 \leq i \leq m} L_i$  is a decomposition of ideals of the Lie algebra  $(k\vec{Q}, [-, -])$ . If non-trivial paths  $\alpha$  and  $\beta$  are sub-paths of a path, then  $\alpha, \beta \in L_i$  for some  $i$ .

**Proof.** By Lemma 2.6(ii) we have  $\alpha \in L_i$  and  $\beta \in L_j$  for some  $i, j$ . By assumption there exists a path  $\gamma$  such that  $\alpha\gamma\beta$  (or  $\beta\gamma\alpha$ ) is a path. Since  $\alpha\gamma\beta = [\alpha, [\gamma, \beta]]$  (or  $\beta\gamma\alpha = [\beta, [\gamma, \alpha]]$ ), it follows that  $\alpha\gamma\beta$  (or  $\beta\gamma\alpha$ ) is contained in  $L_i \cap L_j$ , which implies  $i = j$ .  $\square$

**Lemma 2.9.** Let  $\vec{Q}$  be a finite connected quiver without oriented cycles. Suppose that  $k\vec{Q} = \bigoplus_{1 \leq i \leq m} L_i$  is a decomposition of ideals of the Lie algebra  $(k\vec{Q}, [-, -])$ . If non-trivial paths  $\alpha$  and  $\beta$  lie in a reduced closed walk of the underlying graph  $Q$  of  $\vec{Q}$ , then  $\alpha, \beta \in L_i$  for some  $i$ .

**Proof.** By Lemma 2.6(ii) we may assume that  $\alpha \in L_1$ . Without causing confusion we write  $k\vec{Q} = L_1 \oplus L_2$  with  $L_2 := \bigoplus_{i \geq 2} L_i$ . Let  $\hat{E}_i$  denote the  $k$ -space spanned by all the elements  $e_{t(\gamma)} - e_{s(\gamma)}$ ,  $\gamma \in (\vec{Q}_1 \cap L_i)$ , and  $\hat{E}_i^\perp$  denote the space  $\{x \in k\vec{Q}_0 \mid \langle x, y \rangle = 0, \forall y \in \hat{E}_i\}$  for  $i = 1, 2$ , where  $\langle -, - \rangle$  is defined as in 2.3. Then  $\hat{E}_1^\perp = \{x \in kQ_0 \mid [x, \gamma] = 0, \forall \gamma \in \vec{Q}_1 \cap L_1\}$ . It follows that  $k\vec{Q}_0 \cap L_1 \subseteq \hat{E}_2^\perp$ ,  $k\vec{Q}_0 \cap L_2 \subseteq \hat{E}_1^\perp$ .

By Lemma 2.6(ii) any arrow either lies in  $L_1$ , or lies in  $L_2$ , it follows that  $\hat{E}_1 + \hat{E}_2 = E$ .

Now, if  $\beta \notin L_1$ , then we claim that  $\hat{E}_1 \cap \hat{E}_2 \neq 0$ .

In fact, let  $w$  denote the reduced closed walk. Then  $0 = \sum_{\gamma \in w} \pm(e_{t(\gamma)} - e_{s(\gamma)})$ . Since  $w$  is reduced, it follows that  $0 \neq \sum_{\gamma \in w, \gamma \in L_1} \pm(e_{t(\gamma)} - e_{s(\gamma)}) = \sum_{\gamma \in w, \gamma \in L_2} \pm(e_{t(\gamma)} - e_{s(\gamma)})$ , which proves the claim.

It follows that

$$\dim_k \hat{E}_1 + \dim_k \hat{E}_2 = \dim_k (\hat{E}_1 + \hat{E}_2) + \dim_k (\hat{E}_1 \cap \hat{E}_2) > \dim_k E = \dim_k (k\vec{Q}_0) - 1.$$

By the same argument in 2.3 we have  $\dim_k (\hat{E}_1^\perp) = \dim_k (k\vec{Q}_0) - \dim_k (\hat{E}_1)$  and  $\dim_k (\hat{E}_2^\perp) = \dim_k (k\vec{Q}_0) - \dim_k (\hat{E}_2)$ . Since  $\hat{E}_1 \cap \hat{E}_2 \neq 0$ , it follows that we cannot have  $k\vec{Q}_0 \cap L_1 = \hat{E}_2^\perp$  and simultaneously  $k\vec{Q}_0 \cap L_2 = \hat{E}_1^\perp$ , and hence by Lemma 2.6(i) one gets a contradiction

$$\begin{aligned} \dim_k (k\vec{Q}_0) &= \dim_k (k\vec{Q}_0 \cap L_1) + \dim_k (k\vec{Q}_0 \cap L_2) \\ &< \dim_k (\hat{E}_2^\perp) + \dim_k (\hat{E}_1^\perp) \\ &= \dim_k (k\vec{Q}_0) - \dim_k (\hat{E}_1) + \dim_k (k\vec{Q}_0) - \dim_k (\hat{E}_2) \\ &\leq \dim_k (k\vec{Q}_0). \quad \square \end{aligned}$$

With the preparations above, we can prove the main result in this section. Keep  $P_i, E_i, F_i, 1 \leq i \leq m$ , as in 2.3. For each  $i$ , denote by  $I_i$  the space spanned by  $F_i$  and  $P_i$ .

**Theorem 2.10.** *Let  $\vec{Q}$  be a finite connected quiver without oriented cycles. Then*

$$k\vec{Q} = k \cdot 1 \oplus \bigoplus_{1 \leq i \leq m} I_i$$

is a decomposition into indecomposable ideals of the Lie algebra  $(k\vec{Q}, [-, -])$ .

**Proof.** By Lemma 2.5 and the construction of  $P_i$ , we know that  $k\vec{Q} = k \cdot 1 \oplus \bigoplus_{1 \leq i \leq m} I_i$ . By (2.2) we have  $[F_i, P_j] = 0$  for  $j \neq i$ . Since  $[P_i, P_j] \neq 0$  implies  $i = j$  (by the construction of  $P_i$ 's), it follows that  $[I_i, I_j] = 0$  for  $j \neq i$  and hence  $I_i$  is a Lie ideal. It remains to prove that  $I_i$  is indecomposable.

If  $I_i = V \oplus W$ , then by Lemma 2.6(i) we deduce that  $F_i = (F_i \cap V) \oplus (F_i \cap W)$ . By Lemmas 2.8 and 2.9 we may assume that  $P_i \subseteq V$ . If  $0 \neq x \in F_i \cap W$  then  $[x, V] = 0$ , and hence  $x \in Z(k\vec{Q}) = k \cdot 1$ , which contradicts with  $x \in E$ .  $\square$

**Corollary 2.11.** *Let  $\vec{Q}$  be a finite connected quiver without oriented cycles,  $P_1, \dots, P_m$  be as in 2.2, and  $I_1, \dots, I_m$  be as in Theorem 2.10.*

- (i) *For an arbitrary vector  $(\lambda_1, \dots, \lambda_m) \in k^m$ , there exists a unique inner Poisson structure  $\{-, -\}$  on the path algebra  $k\vec{Q}$  (up to a Poisson algebra isomorphism) such that*

$$\text{ham}(a) = \lambda_i[a, -], \quad \forall a \in I_i, 1 \leq i \leq m.$$

- (ii) *For any vertex  $e$  of  $\vec{Q}$ , the following system of linear equations with variables  $c_1, \dots, c_n$  ( $n := |\vec{Q}_0|$ ), has always a solution:*

$$c_{t(y)} - c_{s(y)} = \lambda_i \omega(y), \quad \forall y \in P_i, 1 \leq i \leq m, \tag{2.3}$$

where  $\omega(y) = 0$  if  $e \neq s(y), e \neq t(y)$ , and  $\omega(y) = 1$  if  $e = t(y)$ , and  $\omega(y) = -1$  if  $e = s(y)$ .

**Proof.** By Theorem 2.10 we can define a linear transformation  $g : k\vec{Q} \rightarrow k\vec{Q}$  by  $g(1) = 0$  and  $g|_{I_i} = \lambda_i \text{Id}$  for each  $1 \leq i \leq m$ . Then  $g$  satisfies (1.1), (1.2): In order to verify this, it suffices to verify that  $g|_{I_i}$  satisfies (1.1) and (1.2), which is clear by construction of  $g$ .

(i) This follows from Lemma 1.2.

(ii) Let  $y \in P_i$ . Assume that  $g(e) = \sum_{i \in \vec{Q}_0} c_i e_i$ . Then by (1.1) we have  $[g(e), y] = [e, g(y)] = \lambda_i [e, y] = \lambda_i \omega(y)y$ . This implies that  $(c_1, \dots, c_m)$  is a solution of the given system of linear equations.  $\square$

### 3. Inner Poisson structures on path algebras

This section is devoted to classifying all the inner Poisson structures on the path algebra  $k\vec{Q}$ , where  $\vec{Q}$  is a finite quiver without oriented cycles. Without loss of generality, we assume that  $\vec{Q}$  is connected.

First we establish two lemmas for the Poisson structures on any associative algebra  $A$  with an orthogonal idempotents decomposition of identity.

**Lemma 3.1.** Let  $(A, \cdot, \{-, -\})$  be a Poisson algebra, and  $1 = \sum_{1 \leq i \leq m} e_i$  with  $e_i e_j = \delta_{i,j} e_i$ ,  $1 \leq i, j \leq m$ .

- (i) We have  $\{e_i, e_j\} = 0, \forall 1 \leq i, j \leq m$ .
- (ii) If  $x \in e_i A e_j, y \in e_p A e_q$ , and  $j \neq p, q \neq i$ , then  $\{x, y\} = e_i e_p \{x, y\} e_q e_j$ .

In particular, if in addition  $i \neq p$ , or  $j \neq q$ , then  $\{e_i A e_j, e_p A e_q\} = 0$ .

**Proof.** This seems to be well known. For the convenience of the reader we include a justification.

- (i) It suffices to prove the assertion for  $i \neq j$ . By the Leibniz rule we have

$$\{e_i, e_j\} = \{e_i^2, e_j\} = e_i \{e_i, e_j\} + \{e_i, e_j\} e_i,$$

and

$$\{e_i, e_j\} = \{e_i, e_j^2\} = e_j \{e_i, e_j\} + \{e_i, e_j\} e_j.$$

It follows that

$$\{e_i, e_j\} = e_i \{e_i, e_j\} e_j + e_j \{e_i, e_j\} e_i.$$

Since  $e_i e_j = 0 = e_j e_i$ , we have

$$0 = \{e_i e_j, e_j\} = e_i \{e_j, e_j\} + \{e_i, e_j\} e_j = \{e_i, e_j\} e_j$$

and

$$0 = \{e_i, e_j e_i\} = e_j \{e_i, e_i\} + \{e_i, e_j\} e_i = \{e_i, e_j\} e_i.$$

Combining the last three identities we have  $\{e_i, e_j\} = 0$ .

- (ii) By the Leibniz rule and (i) we have

$$\begin{aligned} \{x, y\} &= \{e_i x e_j, e_p y e_q\} = e_i \{x e_j, e_p y e_q\} + \{e_i, e_p y e_q\} x \\ &= e_i e_p \{x e_j, y e_q\} + e_i \{x e_j, e_p\} y + e_p \{e_i, y e_q\} x + \{e_i, e_p\} y e_q e_i x \\ &= e_i e_p \{x e_j, y e_q\} + e_i \{x e_j, e_p\} y + e_p \{e_i, y e_q\} x \\ &= e_i e_p x \{e_j, y e_q\} + e_i e_p \{x, y e_q\} e_j + e_i x \{e_j, e_p\} y + e_i \{x, e_p\} e_j e_p y \\ &\quad + e_p y \{e_i, e_q\} x + e_p \{e_i, y\} e_q e_i x \\ &= e_i e_p x \{e_j, y e_q\} + e_i e_p \{x, y e_q\} e_j \\ &= e_i e_p x y \{e_j, e_q\} + e_i e_p x \{e_j, y\} e_q + e_i e_p y \{x, e_q\} e_j + e_i e_p \{x, y\} e_q e_j \\ &= e_i e_p x \{e_j, y\} e_q + e_i e_p y \{x, e_q\} e_j + e_i e_p \{x, y\} e_q e_j \\ &= e_i e_p x e_j e_p \{e_j, y\} e_q + e_i e_p x \{e_j, e_p\} y e_q + e_i e_p y e_q e_i \{x, e_q\} e_j + e_i e_p y \{e_i, e_q\} x e_j \\ &\quad + e_i e_p \{x, y\} e_q e_j \\ &= e_i e_p \{x, y\} e_q e_j. \quad \square \end{aligned}$$

**Lemma 3.2.** *Let  $(A, \cdot, \{-, -\})$  be an inner Poisson algebra, and  $g \in \mathcal{P}(A)$ , such that  $\text{ham}(a) = [g(a), -]$  for each  $a \in A$  (cf. Theorem 1.4). Assume that  $1 = \sum_{1 \leq i \leq m} e_i$  is a decomposition of pairwise orthogonal idempotents. Then  $g(e_i) \subseteq \sum_{1 \leq j \leq m} e_j A e_j$  for all  $i$ .*

**Proof.** By the Pierce decomposition we have  $A = \bigoplus_{1 \leq p, q \leq m} e_p A e_q$ . Write

$$g(e_i) = \sum_{1 \leq p, q \leq m} x_{p,q}$$

with  $x_{p,q} \in e_p A e_q$ . We need to show that  $x_{p,q} = 0$  for  $p \neq q$ . Since  $\{a, b\} = [g(a), b]$  for  $a, b \in A$ , it follows from Lemma 3.1(ii) that

$$0 = \{e_i, e_r\} = [g(e_i), e_r] = \sum_{1 \leq p, q \leq m} [x_{p,q}, e_r] = \sum_{1 \leq p \leq m} x_{p,r} - \sum_{1 \leq q \leq m} x_{r,q}$$

for  $1 \leq r \leq m$ . That is,  $\sum_{1 \leq p \leq m} x_{p,r} = \sum_{1 \leq q \leq m} x_{r,q}$ . It follows that  $x_{p,r} = 0$  for all  $p \neq r$ .  $\square$

Now we turn to the path algebras. Throughout the rest of this section we denote by  $\vec{Q}$  a finite quiver without oriented cycles.

**Lemma 3.3.** *Let  $(k\vec{Q}, \cdot, \{-, -\})$  be an inner Poisson algebra. Then there exists a map  $g \in \mathcal{P}(k\vec{Q})$  such that  $\text{ham}(x) = [g(x), -]$  for each  $x \in k\vec{Q}$ , and that for any non-trivial path  $\alpha$  we have  $g(\alpha) = \lambda_\alpha \alpha$  for some  $\lambda_\alpha \in k$ .*

**Proof.** By Theorem 1.4 there exists a map  $h \in \mathcal{P}(k\vec{Q})$ , such that  $\text{ham}(x) = [h(x), -]$  for each  $x \in k\vec{Q}$ .

By Lemma 3.2 we have  $h(e_i) \in \bigoplus_{j \in \vec{Q}_0} k e_j$  for each vertex  $i$ . It follows that  $\{e_i, \alpha\} = [h(e_i), \alpha] \in k\alpha$ . Write  $h(\alpha) = \lambda_\alpha \alpha + \lambda_1 \alpha_1 + \dots + \lambda_n \alpha_n$  with  $\lambda_1, \dots, \lambda_n \neq 0$ , and  $\alpha, \alpha_1, \dots, \alpha_n$  being pairwise different paths. We claim that there are no non-trivial paths in  $\{\alpha_1, \dots, \alpha_n\}$ . Otherwise, say,  $\alpha_1$  is a non-trivial path. Then  $\{e_{t(\alpha_1)}, \alpha_1\} = \alpha_1$ . Since  $\{e_{t(\alpha_1)}, \alpha_i\} \in k\alpha_i$  for each  $i$ , it follows that we have a contradiction

$$\{e_{t(\alpha_1)}, \alpha\} = [e_{t(\alpha_1)}, h(\alpha)] = [e_{t(\alpha_1)}, \lambda_\alpha \alpha + \lambda_1 \alpha_1 + \dots + \lambda_n \alpha_n] \notin k\alpha.$$

Thus we can write  $h(\alpha) = \lambda_\alpha \alpha + \alpha_0$  with  $\alpha_0 \in k\vec{Q}_0$ , for each non-trivial path  $\alpha$ . Then for any non-trivial path  $\beta$  we have  $\{\alpha, \beta\} = [h(\alpha), \beta] = [\alpha, h(\beta)]$ , that is

$$\lambda_\alpha [\alpha, \beta] + [\alpha_0, \beta] = \lambda_\beta [\alpha, \beta] + [\alpha, \beta_0].$$

Since  $[\alpha_0, \beta] \in k\beta$ ,  $[\alpha, \beta_0] \in k\alpha$ , it follows that  $[\alpha_0, \beta] = 0$  for arbitrary two non-trivial paths  $\alpha$  and  $\beta$ , and hence  $\alpha_0 \in Z(k\vec{Q}) = k \cdot 1$ .

Now define a linear transformation  $g : k\vec{Q} \rightarrow k\vec{Q}$  by

$$g|_{k\vec{Q}_0} = h|_{k\vec{Q}_0}; \quad g(\alpha) = \lambda_\alpha \alpha = h(\alpha) - \alpha_0, \quad \forall \text{ non-trivial path } \alpha.$$

Then  $g \in \mathcal{P}(k\vec{Q})$  since  $h \in \mathcal{P}(k\vec{Q})$ , and  $\text{ham}(x) = [h(x), -] = [g(x), -]$  for each  $x \in k\vec{Q}$ .  $\square$

**Lemma 3.4.** *Let  $g$  and  $\lambda_\alpha$  be as in Lemma 3.3. If  $\alpha$  and  $\beta$  belong to a same  $P_i$  (cf. 2.2), then  $\lambda_\alpha = \lambda_\beta$ .*

**Proof.** First, we claim that if two non-trivial paths  $\alpha$  and  $\beta$  are connected (cf. 2.2), then  $\lambda_\alpha = \lambda_\beta$ . In fact, without loss of generality, we may assume that  $\beta\alpha$  is a path. Then we have

$$\lambda_\alpha\beta\alpha = [\beta, \lambda_\alpha\alpha] = [\beta, g(\alpha)] = \{\beta, \alpha\} = [g(\beta), \alpha] = [\lambda_\beta\beta, \alpha] = \lambda_\beta\beta\alpha.$$

Next, we claim that if two arrows  $\alpha$  and  $\beta$  lie in a reduced closed walk of the underlying graph  $Q$  of  $\vec{Q}$  then  $\lambda_\alpha = \lambda_\beta$ .

In fact, without loss of generality, by the first claim we may assume that each vertex in the reduced closed walk is either a source, or a sink. Thus, it is of the form  $\alpha_1 \cdots \alpha_s$  with  $s \geq 2$ , such that all  $\alpha_i$ 's are arrows and

$$v_1 := s(\alpha_1), \quad v_2 := t(\alpha_1) = t(\alpha_2), \quad \dots, \quad v_s := t(\alpha_s) = t(\alpha_{s-1}), \quad v_1 = s(\alpha_s),$$

and that  $v_1, \dots, v_s$  are pairwise different vertices. We consider  $g(e_{v_1})$ . Set  $\Gamma_0 := \vec{Q}_0 - \{v_1, \dots, v_s\}$ . By Lemma 3.2 we can assume that  $g(e_{v_1}) = \sum_{1 \leq i \leq s} c_i e_{v_i} + z$  with  $z \in \bigoplus_{j \in \Gamma_0} k e_j$ . By (1.1) in Section 1 we have  $[g(e_{v_1}), \alpha_i] = [e_{v_1}, g(\alpha_i)]$  for each  $i$ ,  $1 \leq i \leq s$ . This reads as

$$(c_{t(\alpha_i)} - c_{s(\alpha_i)})\alpha_i = [e_{v_1}, \lambda_{\alpha_i}\alpha_i] = \begin{cases} -\lambda_{\alpha_1}\alpha_1, & \text{if } i = 1; \\ -\lambda_{\alpha_s}\alpha_s, & \text{if } i = s; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$c_1 - \lambda_{\alpha_1} = c_2 = \dots = c_s = c_1 - \lambda_{\alpha_s},$$

and hence  $\lambda_{\alpha_1} = \lambda_{\alpha_s}$ .

Similarly, by considering  $g(e_{v_i})$  we get  $\lambda_{\alpha_i} = \lambda_{\alpha_{i+1}}$ ,  $1 \leq i \leq s - 1$ . In this way we have  $\lambda_{\alpha_1} = \dots = \lambda_{\alpha_s}$ . This completes the proof.  $\square$

Now we are in position to state our main result in this section.

**Theorem 3.5.** *Let  $\vec{Q}$  be a finite connected quiver without oriented cycles, and  $I_1, \dots, I_m$  be as in Theorem 2.10.*

*If  $\{-, -\}$  is an inner Poisson structure on the path algebra  $k\vec{Q}$ , then there exists a unique vector  $(\lambda_1, \dots, \lambda_m) \in k^m$ , such that*

$$\text{ham}(a) = \lambda_i[a, -], \quad \forall a \in I_i, \quad 1 \leq i \leq m. \tag{3.1}$$

*Conversely, for an arbitrary vector  $(\lambda_1, \dots, \lambda_m) \in k^m$ , there exists a unique inner Poisson structure  $\{-, -\}$  on the path algebra  $k\vec{Q}$  (up to a Poisson algebra isomorphism) satisfying (3.1).*

**Proof.** By Corollary 2.11(i) we only need to prove the first assertion.

The uniqueness of  $(\lambda_1, \dots, \lambda_m)$  follows from the fact that  $Z(k\vec{Q}) = k \cdot 1$ . By Lemmas 3.3 and 3.4 we know that there exists a vector  $(\lambda_1, \dots, \lambda_m) \in k^m$ , such that (3.1) is true for  $a \in P_i$ ,  $1 \leq i \leq m$ . It remains to prove that (3.1) is true for  $a \in F_i$  (cf. Theorem 2.10).

By Lemma 1.2 we have a linear transformation  $g : k\vec{Q} \rightarrow k\vec{Q}$  satisfying (1.1), (1.2), and  $g(1) = 0$ , such that  $\text{ham}(x) = [g(x), -]$ ,  $\forall x \in k\vec{Q}$ . It follows that if  $x \in P_i$  then  $g(x) = \lambda_i x + c_i$  with  $c_i \in k \cdot 1$ .

Now let  $a \in F_i$ . If  $y \in P_j, j \neq i$ , then

$$\text{ham}(a)(y) = [g(a), y] = [a, g(y)] = \lambda_j [a, y] = 0 = \lambda_i [a, y].$$

If  $y \in P_i$ , then  $\text{ham}(a)(y) = \lambda_i [a, y]$ . If  $y \in k\vec{Q}_0$  then by Lemma 3.1 we have  $\text{ham}(a)(y) = 0 = \lambda_i [a, y]$ . Thus, in any case (3.1) is true.  $\square$

**Example 3.6.** (i) Theorem 3.5 shows that any inner Poisson structure on a finite-dimensional path algebra is *piecewise standard* (for the definition see Introduction). We point out that in general an inner Poisson structure is not necessarily piecewise standard.

Let  $B$  be a non-commutative  $k$ -algebra, and  $A = B \otimes_k k[x]/\langle x^2 \rangle$ . Consider the following linear map:

$$g : A \rightarrow A, \quad a \mapsto a(1 \otimes x), \quad \forall a \in A.$$

It is easy to check that  $g$  satisfies (1.1) and (1.2). It follows from Lemma 1.2 we have an inner Poisson structure on  $A$  given by  $\text{ham}(a) = [g(a), -]$  for each  $a \in A$ . Note that any element of  $A$  is of the form  $b \otimes 1 + c \otimes x$  with  $b, c \in B$ . If  $A$  is piecewise standard, then there exists an element  $b \otimes 1 + c \otimes x$  with  $b \notin Z(B), c \in B$ , such that

$$\text{ham}(b \otimes 1 + c \otimes x) = [g(b \otimes 1 + c \otimes x), -] = [b \otimes x, -] = \lambda [b \otimes 1 + c \otimes x, -]$$

for some  $\lambda \in k$ . However  $[b \otimes x, b' \otimes 1] \neq \lambda [b \otimes 1 + c \otimes x, b' \otimes 1]$  for some  $b' \in B$ .

(ii) Consider the path algebra  $k\vec{Q}$ , where  $\vec{Q}$  is the quiver with vertices 1, 2, 3, 4, 5, 6, 7, and with arrow  $\alpha$  from 1 to 2, arrow  $\beta$  from 2 to 4, arrow  $\gamma$  from 1 to 3, arrow  $\delta$  from 3 to 4, arrow  $\beta'$  from 5 to 4, arrow  $\alpha'$  from 7 to 5, arrow  $\delta'$  from 6 to 4, arrow  $\gamma'$  from 7 to 6. Let  $I_1$  denote the  $k$ -space spanned by

$$\begin{aligned} & -4e_1 + e_4 + e_5 + e_6 + e_7, \quad -4e_2 + e_4 + e_5 + e_6 + e_7, \quad -4e_3 + e_4 + e_5 + e_6 + e_7 \\ & \alpha, \quad \beta, \quad \gamma, \quad \delta, \quad \beta\alpha, \quad \delta\gamma \end{aligned}$$

and  $I_2$  denote the  $k$ -space spanned by

$$\begin{aligned} & -4e_5 + e_1 + e_2 + e_3 + e_4, \quad -4e_6 + e_1 + e_2 + e_3 + e_4, \quad -4e_7 + e_1 + e_2 + e_3 + e_4 \\ & \alpha', \quad \beta', \quad \gamma', \quad \delta', \quad \beta'\alpha', \quad \delta'\gamma'. \end{aligned}$$

Then  $k\vec{Q} = k \cdot 1 \oplus I_1 \oplus I_2$ , and for arbitrary  $\lambda_1, \lambda_2 \in k$ ,

$$\{x, -\} = \lambda_i [x, -], \quad i = 1, 2,$$

gives an inner Poisson structure on the path algebra  $k\vec{Q}$ ; and any inner Poisson structure on  $k\vec{Q}$  is of this form.

**Remark 3.7.** On the other hand, if  $A$  is an associative algebra and  $1 = \sum_{1 \leq i \leq m} e_i$  with  $e_i e_j = \delta_{i,j} e_i$ ,  $1 \leq i, j \leq m$ , and if in addition  $\dim_k(e_i A e_i) = 1$  for each  $i$ , then any inner Poisson structure on  $A$  is piecewise standard.

In fact, without loss of generality we may assume that  $A$  is indecomposable as an associative algebra. In this case we have  $Z(A) = k \cdot 1$ , and then the assertion follows from a similar argument as in the proof of Theorem 3.5.

As a corollary we see that a Poisson structure on a simple algebra is standard (cf. [Ku1]).

**Remark 3.8.** Let  $(A, \cdot)$  be an associative  $k$ -algebra. In 2.6.1 of Van den Bergh [Van], a Poisson structure on  $(A, \cdot)$  is defined to be a  $k$ -map  $p: A/[A, A] \rightarrow \text{Der}(A)/\text{Inn}(A)$ , such that

$$\{\bar{a}, \bar{b}\}_p := \overline{(p(a))^{-1}(b)} \in A/[A, A]$$

is a Lie bracket on  $A/[A, A]$ , where  $(p(a))^{-1}$  is an arbitrary lift of  $p(a)$ . If  $(A, \cdot)$  is commutative then this definition is equivalent to the one we used in this paper. But it is different from the one we used in non-commutative case (examples can be easily given by Theorem 3.5). We are indebted to Crawley-Boevey for pointing out this to us. If  $p$  is a Poisson structure on  $(A, \cdot)$  in the sense above, then  $\{-, -\}$  induces naturally a Poisson structure on  $A/\langle [A, A] \rangle$ , where  $\langle [A, A] \rangle$  is the ideal of  $A$  generated by  $[A, A]$ .

Another type of non-commutative Poisson structure on  $(A, \cdot)$ , which is slightly different from the one above has been given more recently in Crawley-Boevey [C]. For details see Section 1 in [C].

#### 4. Quivers admitting outer Poisson structures

By an *outer Poisson algebra* we mean a Poisson algebra which is not inner. By an *outer-Poisson quiver* we mean a finite quiver  $\vec{Q}$  without oriented cycles, such that there exists an outer Poisson structure on the path algebra  $k\vec{Q}$ . A finite quiver  $\vec{Q}$  without oriented cycles is said to be an *inner-Poisson quiver* provided that it is not an outer-Poisson quiver, i.e. any Poisson structure on the path algebra  $k\vec{Q}$  is an inner Poisson algebra.

The aim of this section is to classify all the inner-Poisson quivers, and hence all the outer-Poisson quivers.

A theorem of Happel [H] says that the first Hochschild cohomology of a finite-dimensional path algebra  $k\vec{Q}$  vanishes if and only if  $\vec{Q}$  is a finite tree. It follows that a finite tree is an inner-Poisson quiver. However the converse is not true (see Example 4.3(i) below).

It is easy to see that an inner-Poisson quiver is a disjoint union of connected inner-Poisson quivers. It follows that we may assume that the quivers considered are connected.

**4.1.** We fix some notations. Let  $\vec{Q}$  be a finite connected quiver without oriented cycles. For two non-trivial paths  $\alpha$  and  $\beta$ , define  $\alpha \approx \beta$  if and only if there exist non-trivial paths  $\gamma_0 = \alpha, \gamma_1, \dots, \gamma_t, \gamma_{t+1} = \beta$ , such that for each  $0 \leq i \leq t, \gamma_i$  and  $\gamma_{i+1}$  are connected (cf. 2.2). It is clear that  $\approx$  is an equivalence relation on the set of non-trivial paths of  $\vec{Q}$ . Let  $\{G_1, \dots, G_s\}$  be the set of the equivalence classes. By definition if  $\alpha \approx \beta$  then  $\alpha \not\approx \beta$ .

**Theorem 4.2.** Let  $\vec{Q}$  be a finite connected quiver without oriented cycles. Then  $\vec{Q}$  is an inner-Poisson quiver if and only if the relation  $\approx$  is exactly the relation  $\not\approx$ , i.e. the following condition holds:



if two arrows  $\alpha$  and  $\beta$  lie in a reduced closed walk of the underlying graph  $Q$  of  $\vec{Q}$ ,  
 then  $\alpha \approx \beta$ . (4.1)

**Example 4.3.** (i) Let  $\vec{Q}$  be the quiver with vertices 1, 2, 3, 4, 5, and with one arrow from 1 to 2, one arrow from 2 to 4, one arrow from 1 to 3, one arrow from 3 to 4, and one arrow from 4 to 5. Then by Theorem 4.2  $\vec{Q}$  is an inner-Poisson quiver, but the first Hochschild cohomology group is of dimension one, by Proposition 1.6 in [H].

(ii) Let  $\vec{Q}$  be the quiver with vertices 1, 2, 3, 4, and with one arrow from 1 to 2, one arrow from 2 to 4, one arrow from 1 to 3, one arrow from 3 to 4. Then by Theorem 4.2  $\vec{Q}$  is an outer-Poisson quiver.

For the proof of Theorem 4.2 we first give some lemmas.

**Lemma 4.4.** Let  $\vec{Q}, \{G_i\}_{1 \leq i \leq s}$  be as in 4.1,  $\lambda_1, \dots, \lambda_s \in k$ . Define

$$\{x, y\} = \begin{cases} 0, & \text{if } x, y \in \vec{Q}_0, \\ \lambda_i[x, y], & \text{if } x \in G_i, \text{ or } y \in G_i. \end{cases}$$

Then  $(k\vec{Q}, \cdot, \{-, -\})$  is a Poisson algebra.

**Proof.** Left to the reader.  $\square$

**Lemma 4.5.** Let  $\vec{Q}, \{G_i\}_{1 \leq i \leq s}$  be as in 4.1. Let  $\lambda_1, \dots, \lambda_s$  be pairwise different elements in  $k$ . Suppose that there exist  $\alpha \in G_i$  and  $\beta \in G_j$  with  $i \neq j$ , such that  $\alpha$  and  $\beta$  lie in a reduced closed walk of the underlying graph  $Q$  of  $\vec{Q}$ , then the Poisson structure given in Lemma 4.4 is not inner.

**Proof.** Let  $\alpha \in P_t$  for some  $t, 1 \leq t \leq m$ , where  $P_t$  is defined in 2.2. Then  $\beta \in P_t$ . If the Poisson algebra in Lemma 4.4 is inner, then by Theorem 3.5 there exists a unique vector  $(c_1, \dots, c_m) \in k^m$  such that  $\text{ham}(\alpha) = [c_t\alpha, -]$ ,  $\text{ham}(\beta) = [c_t\beta, -]$ . On the other hand, by definition we have  $\text{ham}(\alpha) = [\lambda_i\alpha, -]$ ,  $\text{ham}(\beta) = [\lambda_j\beta, -]$ . It follows that we get a contradiction  $\lambda_i = c_t = \lambda_j$ .  $\square$

**Lemma 4.6.** Let  $\vec{Q}$  be a finite quiver without oriented cycles satisfying the condition (4.1), and  $\{G_i\}_{1 \leq i \leq s}$  be as in 4.1. Assume that  $(k\vec{Q}, \cdot, \{-, -\})$  is a Poisson algebra. Then there exist  $\lambda_1, \dots, \lambda_s \in k$ , such that  $\{\alpha, e_{t(\alpha)}\} = \{e_{s(\alpha)}, \alpha\} = \lambda_i[\alpha, e_{t(\alpha)}] = \lambda_i[e_{s(\alpha)}, \alpha]$  for  $\alpha \in G_i, 1 \leq i \leq s$ .

**Proof.** First, we claim that it suffices to prove the assertion for any arrow  $\alpha \in G_i, 1 \leq i \leq s$ .

In fact, suppose that the assertion holds for any arrow. Let  $\gamma = \beta\alpha$  be a non-trivial path with  $\beta$  an arrow in  $G_i$  and  $\alpha$  a non-trivial path. Then by the Leibniz rule and Lemma 3.1(ii) we have

$$\begin{aligned} \{\gamma, e_{t(\gamma)}\} &= \beta\{\alpha, e_{t(\gamma)}\} + \{\beta, e_{t(\gamma)}\}\alpha \\ &= \{\beta, e_{t(\beta)}\}\alpha = \lambda_i[\beta, e_{t(\beta)}]\alpha \\ &= \lambda_i[\gamma, e_{t(\gamma)}]. \end{aligned}$$

Similarly we have  $\{e_{s(\gamma)}, \gamma\} = \lambda_i[e_{s(\gamma)}, \gamma]$ . This proves the claim.

Now, we claim that for any arrow  $\alpha$  there exists  $\lambda_\alpha \in k$  such that  $\{\alpha, e_{t(\alpha)}\} = \{e_{s(\alpha)}, \alpha\} = \lambda_\alpha[\alpha, e_{t(\alpha)}] = \lambda_\alpha[e_{s(\alpha)}, \alpha]$ .

In fact, since  $\vec{Q}$  has no oriented cycles, it follows from the Leibniz rule and Lemma 3.1(i) that

$$\begin{aligned} \{\alpha, e_{t(\alpha)}\} &= e_{t(\alpha)}\{\alpha, e_{t(\alpha)}\} + \{e_{t(\alpha)}, e_{t(\alpha)}\}\alpha \\ &= e_{t(\alpha)}\{\alpha, e_{t(\alpha)}\}e_{s(\alpha)} + \alpha\{e_{s(\alpha)}, e_{t(\alpha)}\} \\ &= e_{t(\alpha)}\{\alpha, e_{t(\alpha)}\}e_{s(\alpha)}. \end{aligned}$$

Similarly we have  $\{e_{s(\alpha)}, \alpha\} = e_{t(\alpha)}\{\alpha, e_{t(\alpha)}\}e_{s(\alpha)}$ . It follows that  $\{\alpha, e_{t(\alpha)}\} = \{e_{s(\alpha)}, \alpha\} = \sum_{1 \leq i \leq r} c_i \alpha_i$ , where  $\alpha_i$ 's are pairwise different paths with  $s(\alpha_i) = s(\alpha)$  and  $t(\alpha_i) = t(\alpha)$  for all  $i$ . If  $r = 1$  and  $\alpha_1 = \alpha$  then the claim is proved. Otherwise,  $\alpha$  and  $\alpha_i$  form a reduced closed walk, and hence by the condition (4.1) we have  $\alpha \approx \alpha_i$  for each  $i$ . While  $\alpha$  is an arrow, this implies that there exists an arrow  $\beta$  such that  $s(\beta) = t(\alpha)$ , or  $t(\beta) = s(\alpha)$ . We only discuss the case that  $s(\beta) = t(\alpha)$ . The other case can be treated in the same way.

Write  $\{\beta, e_{t(\beta)}\} = \{e_{s(\beta)}, \beta\} = \sum_{1 \leq j \leq n} d_j \beta_j$ , where  $t(\beta_j) = t(\beta)$  and  $s(\beta_j) = s(\beta) = t(\alpha)$  for all  $j$ . By the Leibniz rule and Lemma 3.1(i) we have

$$\begin{aligned} \{\alpha, \beta\} &= \{\alpha, \beta\}e_{s(\beta)} + \beta\{\alpha, e_{t(\alpha)}\} \\ &= \alpha\{e_{s(\alpha)}, \beta\}e_{s(\beta)} + \beta\{\alpha, e_{t(\alpha)}\} \\ &= \alpha e_{t(\beta)}\{e_{s(\alpha)}, \beta\}e_{s(\beta)} + \beta\{\alpha, e_{t(\alpha)}\} \\ &= \beta\{\alpha, e_{t(\alpha)}\} = \sum_{1 \leq i \leq r} c_i \beta \alpha_i. \end{aligned}$$

By applying the Leibniz rule on the opposite side one gets

$$\{\alpha, \beta\} = \{e_{t(\alpha)}, \beta\}\alpha = \{e_{s(\beta)}, \beta\}\alpha = \sum_{1 \leq j \leq n} d_j \beta_j \alpha.$$

Thus

$$\sum_{1 \leq i \leq r} c_i \beta \alpha_i = \sum_{1 \leq j \leq n} d_j \beta_j \alpha.$$

This implies  $\alpha_i = \alpha$  for all  $i$  with  $1 \leq i \leq r$  (otherwise one gets an oriented cycles). This contradicts the assumption, and hence the claim is proved.

The argument above also proves that if  $\alpha$  and  $\beta$  are arrows such that  $s(\beta) = t(\alpha)$  then  $\lambda_\alpha = \lambda_\beta$ . It follows from the definition of  $G_i$ 's that  $\lambda_\alpha = \lambda_\beta$  if  $\alpha$  and  $\beta$  belong to a same  $G_i$ . This completes the proof.  $\square$

**Lemma 4.7.** *Let  $\vec{Q}$  be a finite quiver without oriented cycles satisfying the condition (4.1), and  $\{G_i\}_{1 \leq i \leq s}$  be as in 4.1. Assume that  $(k\vec{Q}, \cdot, \{-, -\})$  is a Poisson algebra, and  $\lambda_1, \dots, \lambda_s \in k$ , such that  $\{\alpha, e_{t(\alpha)}\} = \{e_{s(\alpha)}, \alpha\} = \lambda_i[\alpha, e_{t(\alpha)}]$  for  $\alpha \in G_i$ ,  $1 \leq i \leq s$ . Then for each  $\alpha \in G_i$ ,  $1 \leq i \leq s$ , we have  $\{\alpha, \gamma\} = \lambda_i[\alpha, \gamma]$  for arbitrary non-trivial path  $\gamma$ .*

**Proof.** First, we claim that it suffices to prove the assertion for any arrow  $\alpha$ .

In fact, suppose that the assertion holds for any arrow. Let  $\beta\alpha$  be an non-trivial path with  $\beta$  and  $\alpha$  both non-trivial paths in  $G_i$ . Then by induction on the length of path we have

$$\{\beta\alpha, \gamma\} = \beta\{\alpha, \gamma\} + \{\beta, \gamma\}\alpha = \lambda_i\beta\{\alpha, \gamma\} + \lambda_i[\beta, \gamma]\alpha = \lambda_i[\beta\alpha, \gamma].$$

So, we may assume that  $\alpha$  is an arrow in  $G_i$ . We prove the assertion case by case.

*Case 1:* If  $t(\alpha) = s(\gamma)$ , then by the Leibniz rule we have  $\{\alpha, \gamma\} = \alpha\{e_{s(\alpha)}, \gamma\} + \{\alpha, \gamma\}e_{s(\alpha)}$ . Since  $\vec{Q}$  has no oriented cycles, it follows that  $s(\alpha) \neq s(\gamma)$  and  $s(\alpha) \neq t(\gamma)$ . It follows from Lemma 3.1(i) that

$$\{\alpha, \gamma\}e_{s(\gamma)} = \alpha\{e_{s(\alpha)}, \gamma\}e_{s(\gamma)} + \{\alpha, \gamma\}e_{s(\alpha)}e_{s(\gamma)} = \alpha\{e_{s(\alpha)}, \gamma\}e_{s(\gamma)} = \alpha e_{t(\gamma)}\{e_{s(\alpha)}, \gamma\}e_{s(\gamma)} = 0,$$

and hence

$$\{\alpha, \gamma\} = \gamma\{\alpha, e_{s(\gamma)}\} + \{\alpha, \gamma\}e_{s(\gamma)} = \gamma\{\alpha, e_{t(\alpha)}\} = \lambda_i\gamma\{\alpha, e_{t(\alpha)}\} = \lambda_i[\alpha, \gamma].$$

*Case 2:* If  $s(\alpha) = t(\gamma)$ , then by a similar argument as in Case 1 we have  $\{\alpha, \gamma\} = \{\alpha, e_{s(\alpha)}\}\gamma = \lambda_i[\alpha, \gamma]$ .

*Case 3:* If  $t(\alpha) \neq s(\gamma)$ ,  $s(\alpha) \neq t(\gamma)$ ,  $t(\alpha) \neq t(\gamma)$ , then by Lemma 3.1(ii)  $\{\alpha, \gamma\} = 0 = \lambda_i[\alpha, \gamma]$ .

*Case 4:* If  $t(\alpha) \neq s(\gamma)$ ,  $s(\alpha) \neq t(\gamma)$ ,  $s(\alpha) \neq s(\gamma)$ , then by Lemma 3.1(ii)  $\{\alpha, \gamma\} = 0 = \lambda_i[\alpha, \gamma]$ .

*Case 5:* The unique case left is  $t(\alpha) = t(\gamma)$ ,  $s(\alpha) = s(\gamma)$ . In this case  $\alpha$  and  $\gamma$  form a reduced closed walk, and hence  $\gamma \in G_i$  by (4.1). Then by assumption and  $\alpha\gamma = 0$  we have

$$\begin{aligned} \{\alpha, \gamma\} &= \alpha\{e_{s(\gamma)}, \gamma\} + \{\alpha, \gamma\}e_{s(\alpha)} \\ &= \lambda_i\alpha\{e_{s(\gamma)}, \gamma\} + \{\alpha, e_{t(\alpha)}\}\gamma e_{s(\alpha)} + e_{t(\alpha)}\{\alpha, \gamma\}e_{s(\alpha)} \\ &= \lambda_i\alpha\{e_{s(\gamma)}, \gamma\} + \lambda_i[\alpha, e_{t(\alpha)}]\gamma + e_{t(\alpha)}\{\alpha, \gamma\}e_{s(\alpha)} \\ &= e_{t(\alpha)}\{\alpha, \gamma\}e_{s(\alpha)}. \end{aligned}$$

Since  $\alpha$  and  $\gamma$  form a reduced closed walk and  $\alpha$  is an arrow, it follows from (4.1) that there exists an arrow  $\beta$  such that  $s(\beta) = t(\alpha)$ , or  $t(\beta) = s(\alpha)$ . Without loss of generality we may assume that  $s(\beta) = t(\alpha)$ . Then by Case 3 we have  $\{\alpha, \beta\gamma\} = 0$ . By Case 1 we have  $\{\alpha, \beta\}\gamma = \lambda_i[\alpha, \beta]\gamma = 0$ . It follows that

$$0 = \{\alpha, \beta\gamma\} = \beta\{\alpha, \gamma\} + \{\alpha, \beta\}\gamma = \beta\{\alpha, \gamma\}.$$

While  $\beta\{e_{s(\beta)}\{\alpha, \gamma\}\} = \beta\{\alpha, \gamma\} = 0$  implies that  $e_{s(\beta)}\{\alpha, \gamma\} = 0$ , i.e.  $\{\alpha, \gamma\} = 0 = \lambda_i[\alpha, \gamma]$ .

This completes the proof.  $\square$

**4.8. Proof of Theorem 4.2.** By Lemma 4.5 we only need to show the “if” part.

Assume that the condition (4.1) is satisfied. Let  $(k\vec{Q}, \cdot, \{-, -\})$  be a Poisson algebra. We need to prove that there exists a map  $g : \bigcup_{n \geq 0} \vec{Q}_n \rightarrow \bigcup_{n \geq 0} \vec{Q}_n$ , such that  $\text{ham}(x) = \{g(x), -\}$ ,  $\forall x \in \bigcup_{n \geq 0} \vec{Q}_n$ .

For  $\alpha \in G_i$ , define  $g(\alpha) = \lambda_i \alpha$ , where  $\lambda_i$  is given as in Lemma 4.6. Then by construction, Lemmas 3.1(ii) and 4.7 we have

$$\begin{aligned} \text{ham}(\alpha)(y) = \{\alpha, y\} &= \begin{cases} \lambda_i[\alpha, y], & \text{if } y = e_t(\alpha), \text{ or } y = e_s(\alpha); \\ 0, & \text{if } y \in \vec{Q}_0, y \neq e_t(\alpha), y \neq e_s(\alpha); \\ \lambda_i[\alpha, y], & \text{if } y \in \bigcup_{n \geq 1} \vec{Q}_n \end{cases} \\ &= [\lambda_i \alpha, y] = [g(\alpha), y]. \end{aligned}$$

For  $\alpha = e \in \vec{Q}_0$ , define  $g(e) = \sum_{i \in \vec{Q}_0} c_i e_i$ , where  $(c_1, \dots, c_m)$  is a solution of the system of linear equations (2.3) (cf. Corollary 2.11(ii)). Then by construction and Lemmas 3.1 and 4.6 we have  $\text{ham}(e) = [g(e), -]$ . This completes the proof.  $\square$

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